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A NOTE ON THE ALGEBRA OF QUALITATIVE EQUIVALENCE OF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

The notion of qualitative equivalence of ordinary differential equations is a concept which assists in describing the behavior of an equation without necessarily drawing the solution curves of the equation. This is achieved via the geometrical representation of the equation. Qualitative equivalence of a set of differential equations necessarily results into the generation of the qualitative classes of the set – a discrete structure. Practical areas of application of this concept include computer networking, fractals and coding theory. In this paper, the basic principle of the idea is presented using a simple illustration, namely, the qualitative equivalence behavior of the equation $x^/ = x(1-a) + b$, where a, b Є IR, from the point of view of the equilibrium point of the equation. It is shown that the phase portrait of the set $S = \{x^/ = x(1-a) + b : a, b \in IR\}$ of first order autonomous ordinary differential equations is an attractor if $a > 0$ and a repellor if $a \leq 0$. It is also shown that S has two qualitative classes. Furthermore, qualitative equivalence is established between some subsets of S and some subsets of the universal set $L = \{ x^/ = ax + b : a, b \in IR\}$ of first order linear autonomous ordinary differential equations

Keywords and phrases: Qualitative equivalence, first order autonomous ordinary differential equations, qualitative classes, phase portrait, equilibrium point.

I. INTRODUCTION

The notion of qualitative equivalence of ordinary differential equations is a concept which assists in describing the behavior of an equation without necessarily drawing the solution curves of the equation. This is achieved via the geometrical representation of the equation. Qualitative equivalence of a set of differential equations necessarily results into the generation of the qualitative classes of the set – a discrete structure. Practical areas of application of this concept include computer networking and coding theory. In this paper, the author studies the qualitative equivalence characteristics of the solutions of first order autonomous ordinary differential equation $x' = x(1 – a) + b$, where $a, b \in IR$ from the point of view of the equilibrium point of the equation. The paper is based on the existence and uniqueness properties of solutions.

Let $f : IR \rightarrow IR$ be a continuous function. Consider the general first order autonomous equation

$$x' = f(x) \qquad (1.1)$$

We seek to describe the behavior of the solutions of (1.1) without necessarily drawing the solution curves of the equation. The approach is to give a geometrical representation of the qualitative behavior of the equation [1]. This representation is called the phase portrait or phase diagram. Of central importance in drawing the phase portrait is the knowledge of the equilibrium point of (1.1). An equilibrium point (or critical point or singular point or stationary point) of (1) is the point c such that $f(c) = 0$. The phase portrait of any equation of the form (1.1) is completely determined by the nature of its equilibrium points. By the Fundamental Theorem of Algebra and its Corollary [2], the maximum number of equilibrium points of (1.1) is the same as the degree of $f(x)$. In general, the phase portrait of an equilibrium point c of (1.1) can only be one of the following viz attractor, repellor, positive shunt and negative shunt. These are called the qualitative types [4, 6].

II. MAIN RESULTS

This section contains the main results of the paper

Definition 2.1 [4]

Let

$$x' = f(x) \qquad (2.1A)$$

and

$$y' = g(y) \qquad (2.1B)$$

be two first order autonomous ordinary differential equations. Then the two equations are said to be qualitatively (or topologically) equivalent if there exists a continuous bijection from the phase portrait of (2.1A) onto the phase portrait of (2.1B) in such a way that the orientation of the phase portraits is preserved.

Theorem 2.2 [4]

Let E_1 and E_2 be two first order autonomous ordinary differential equations. Then the two equations are qualitatively equivalent if they have the same qualitative type.

Proof

Suppose E_1 and E_2 have the same phase portrait. Then an orientation preserving continuous bijections can be defined from the phase portrait of E_1 onto that of E_2. Therefore E_1 and E_2 are qualitatively equivalent. □

Theorem 2.3 [4]

Suppose E_1 and E_2 are two first order autonomous ordinary differential equations. Let $E_1 \sim E_2$ iff E_1 is qualitatively equivalent to E_2. Then $\sim$ defines an equivalence relation.

Proof

This follows from the fact that $\sim$ is reflexive, symmetric and transitive □

We now consider the set

$L = \{x' = f(x) \mid f: IR \rightarrow IR\}$

of first order autonomous ordinary differential equations, where IR is the set of real numbers. By the Fundamental Theorem of Equivalence Relations [2, 3], ˜partitions L into disjoint equivalence classes called qualitative classes. All equations which belong to the same qualitative class exhibit the same qualitative behavior. This behavior gives a description of the existence and uniqueness properties of the solutions of (1.1). The qualitative behavior is essentially determined by the function f.

Theorem 2.4

Let

$S = \{x' = f(x) = x(1 - a) + b : a, b \in IR\}$

be a set of first order autonomous ordinary differential equations. Then

(i) The phase portrait of S is an attractor if $a > 0$ and it is a repellor if $a \leq 0$.

(ii) S has two qualitative classes.

Proof

(i) We define the following subsets of S :

$S_1 = \{x' = x(1 - a) + b : a > 0, b > 0\}$

$S_2 = \{x' = x(1 - a) + b : a < 0, b > 0\}$

$S_3 = \{x' = x(1 - a) + b : a > 0, b < 0\}$

$S_4 = \{x' = x(1- a) + b : a < 0, b < 0\}$

$S_5 = \{x' = x(1 - a) + b : a = 0, b > 0\}$

$S_6 = \{x' = x(1 - a) + b : a = 0, b < 0\}$

$S_7 = \{x' = x(1 - a) + b : a > 0, b = 0\}$

$S_8 = \{x' = x(1 - a) + b : a < 0, b = 0\}$

$S_9 = \{x' = x(1-a) + b: a = b = 0\}$

where $S = \cup S_i$ and $S_i \cap S_j = \emptyset$ for all $1 \le i \le 9$, $1 \le j \le 9$, $i \ne j$. We note that the equilibrium point of S is –b/(1-a). Consider S_1, S_3 and S_7 which are subsets of S in which a > 0. When x > -b/(1-a) (or < -b/(1-a)),f(x) < 0 (or > 0) respectively, where a ≠ 1. The phase portrait of each of S_1, S_3 and S_7 is thus an attractor. Suppose a ≤ 0 and consider S_2, S_4, S_5, S_6, S_8 and S_9.When x > -b(1-a) (or < -b(1-a)), f(x) > 0 (or < 0) respectively. Therefore, the phase portrait of each of these subsets is a repellor. Hence the result. □

(ii) That S has two qualitative classes follows from the fact that every subset of S is either an attractor or a repellor. □

Theorem 2.5

Let

$S^* = \{x' = x(1-a) + b : a > 0, a, b \in IR\}$

$L^* = \{x' = ax + b : a < 0, a, b \in IR\}$

$P^* = \{x' = x(1-a) + b : a \le 0, a, b \in IR\}$

$Q^* = \{x' = ax + b : a > 0, a, b \in IR\}$

be sets of first order linear autonomous ordinary differential equations, where IR is the set of real numbers. Then S* is qualitatively equivalent to L* and P* is qualitatively equivalent to Q*.

Proof

It shall be shown that S* and L* have the same qualitative type and P* and Q* have the same qualitative type. By Theorem 2.4, the phase portrait of S* is an attractor. For L*, f(x) < 0 (or > 0) when x > - b/a (or < - b/a) respectively, and so the phase portrait of L* is an attractor. Also by Theorem 2.4, the phase portrait of P* is a repellor. Now with respect to Q*, f(x) > 0 (or < 0) when x > - b/a (or < -b/a) respectively, and so the phase portrait of Q* is also a repellor. The theorem is thus proved. □

III. DISCUSSION AND CONCLUSION

In this paper, the notion of qualitative equivalence has been applied to the set $S = \{x' = x(1-a) + b : a, b \in IR\}$, where IR is the set of real numbers. This set is a subset of the universal set $L = \{x' = ax + b : a, b \in IR)$ of first order linear autonomous ordinary differential equations. It is shown that S has two qualitative classes. It is then proved that certain subsets of L are qualitatively equivalent to some subsets of S. The paper gives a description of the qualitative behavior of the equation $x' = x(1-a) + b$, where $a, b \in IR$. Practical application of the notion of qualitative equivalence of a set include computer networking, fractals and coding theory [5, 6, 7].

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