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ON SOME COMBINATORIAL RESULTS OF COLLAPSE AND PROPERTIES OF HEIGHT IN FULL TRANSFORMATION SEMIGROUPS

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ABSTRACT

In a Transformation Semigroup S, an element α in S is collapsible, c(α) if there exists a number q= $|t\alpha^{-1}|$ **≥ 2 where t is an element in the image of α. In this note we obtain formula for the total number of collapsible elements for** $|t\alpha^{-1}|$ **=2 and** $|t\alpha^{-1}|$ **=3 for n ≥ 2 in** T_n **, the full transformation semigroup. Also q is defined for some height h (α) =**$|\mathrm{Im}\,\alpha|$**.**

Keywords: Collapse, Height, Full Transformation Semigroup, Idempotent, Semigroup

I INTRODUCTION

Among 'naturally occurring' semigroups, the full transformation semigroup $T = T_x$, consisting of all self-maps of a non-empty set $X = X_n$, has been perhaps the most intensively studied[7]. For standard terms in semigroup theory see, for example,[4].

A groupoid (S, μ) is defined as a non-empty set S on which a binary operation $, \mu$– by which we mean a map $\mu : S \times S \rightarrow S$– is defined. We say that (S, μ) is a Semigroup if the operation μ is associative, that is to say, if, for all x, y and z in S,

$$((x, y)\mu, z)\mu = (x, (y, z)\mu)\mu .$$

Let $X_n = \{1, 2, 3, . . ., n\}$. Then a transformation

$$\alpha: \mathrm{Dom}\alpha \subseteq X_n \rightarrow \mathrm{Im}\,\alpha$$

is said to be full or total if $\mathrm{Dom}\alpha = X_n$.

The full transformation semigroup is also known as the symmetric semigroup. It is also a well known fact that, if $S = T_n$, then $|S| = n^n$ [1]. Enumerative problems of an essentially combinatorial in nature arise naturally in the study of semigroups of transformations [2]. An element $\alpha \in T_n$ is an idempotent, if α2 = α. If we denote by S_n the symmetric group, which is one – one in nature , that is $S_n \subseteq T_n$, then Howie [3] showed that $T_n \backslash S_n$ = Sin g_n is idempotent generated and Tainiter [6] showed that the total number of its idempotents is

$$\sum_{r=1}^{n-1} \binom{n}{r} r^{n-r}$$

Another study on the full transformation semigroup which is of interest in this paper is knowing the total number of collapsible elements, |c (α)|. Howie [3] defined a collapsible element as:

$$c(\alpha) = \bigcup_{t \in \mathrm{Im}\,\alpha} \{t\alpha^{-1} : |t\alpha^{-1}| \geq 2\}$$

In his paper, Howie [5] still defined collapsible element as

$$c(\alpha) = \sum_{t \in \mathrm{Im}\,\alpha} (|\ t\alpha^{-1}\ |) - 1$$

The former is of interest in this paper.

We shall denote the number of collapsible elements by

$$\eta = |\ c(\alpha)\ | = |\ \bigcup_{t \in \mathrm{Im}\,\alpha} \{t\alpha^{-1} : |t\alpha^{-1}| \geq 2\}\ |$$

For the non-collapsible elements of T_n, which is the symmetric group S_n, we have the expression, n! for c(α) = 0. That is, a situation when no image appears more than once. The number of collapsible elements in T_n does not seem to have been investigated before and we set out to investigate these numbers. In this paper we obtain formula for the total number of collapsible elements for $|t\alpha^{-1}|$ =2 and $|t\alpha^{-1}|$ =3 for n ≥ 2 in T_n , the full transformation semigroup. Also q is defined for some height h (α) = $|\mathrm{Im}\,\alpha|$.

II COMBINATORIAL RESULTS

Let S be the semigroup T_n . If $\alpha \in S$, h(α) = $|\mathrm{Im}\,\alpha|$, the length of the image of α . Then the principal result of this paper is:

Theorem 1:

Let $X_n = \{1, 2, 3 . . . n\}$, $\alpha \in S$ and $\eta = |\ \bigcup_{t \in \mathrm{Im}\,\alpha} \{t\alpha^{-1} : |t\alpha^{-1}| \geq 2\}\ |$

Then with the initial value of n as the value of |tα-1|,

$\eta = n!\binom{n}{q}$ for q = |tα-1| = 2 and

$\eta = \frac{n}{2}\binom{n}{q}$ for q = 3.

As a step towards proving the theorem above, we have the following :

<u>Lemma 1</u>: An element α with h (α) = n is not collapsible.

<u>Proof</u>: Let $\alpha \in S$. Let Imα = tα for each $t \in X_n$ and domain of α is n i.e, domα = |tα-1| = n. Assume that |Imα| ≠ 1, it means that |tα| ≥ 2 and it implies that the domain |tα-1| ≥ 2.

Taking |tα| = 1 does not imply |tα-1| = 1. Infact |tα-1| cannot be 1 inorder to satisfy collapsible property. Thus |tα-1| ≥ 2 and |tα| ≥ 1.

$E_{24}(z)$

Now let |tα-1| = n then we have what is called a symmetric group, S_n which is one-one and not collapsible. Hence the result η = n! for q = 0▪.

<u>Proof of Theorem 1</u>

Let $t_k \in \text{Im}\, \alpha$, S = T_n and dom α = |tα-1| = n. Let $\alpha_y \in S$ be such that

$$\alpha_y = \begin{pmatrix} 1 & 2 & 3... & n \\ t_1 & t_2 & t_3 & t_k \end{pmatrix}$$

For α_y to be collapsible, $t_k \leq n$ and $|t_k\, \alpha| < n$ where y represents each element in the order of T_n which is nn in number. That is, we have $\alpha_y = \alpha_1, \alpha_2, \alpha_3 ... \alpha_{n^n}$ If dom α = n and $|t_k\, \alpha| = 2$ for a particular $\alpha_y \in S$,
$\Rightarrow$ h(α) = |Im α| = 2 and in that case $|t_k\, \alpha\text{-}1| > 2$. From lemma 1,h(α) ≠ n for a collapsible element with domain n. Thus for n= 2, |tα-1| = 2 $\Rightarrow$ h(α) = 1. If $|t_k\, \alpha\text{-}1| > 2$ then $|t_k\, \alpha| \geq 1$. Therefore for $|t_k\, \alpha\text{-}1|$ = 2= q , we have $\eta = n!\binom{n}{q}$ and for $|t_k\, \alpha\text{-}1|$ = 3,

$$\eta = \frac{n!}{2}\binom{n}{q}$$

with the condition that n starts from the value of q▪

III CONCLUSION

It can be deduced from thorem 1 and lemma 1 that if
q= n, then h(α) = 1;
q = 2, then h(α) = n-1 and when
q = 3 then h(α) = n-2 ∀ n. This is to establish the relationship between collapse and height in this paper. Also, values of η is known for each of q = 2 and q = 3.

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