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ON SET-THEORETIC FUNDAMENTALS OF SYSTEMS INTEGRATION

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ABSTRACT

In spite of the much touted need to achieve systems integration, very little has been accomplished in practice. This is often so because there is no analytical basis for identifying and modeling the parameters that the sub-systems have in common. Consequently, attempts to achieve systems integration degenerates to mere conceptual formulations. Sometimes the conceptual formulations work, but most times they do not. The premise of this paper is that a rigorous set-theoretic formulation can help identify the dominant common elements of subsystems, thereby facilitating practical implementation that is amenable to quantitative optimization. The paper explores fundamental set theory and principles for modeling systems interactions.

Keywords: Systems Integration, Set Theory, Mathematical Modeling, Sub-system Analysis, Surface Project Integrals, Quantitative optimization

0. INTRODUCTION

In order for two systems to be integrated, they must overlap in scope or performance with respect to time, space, or physics; subsequently, this requires mathematical representations for manipulating the integrated parameters of the systems. In spite of the much touted need to achieve systems integration, very little has been accomplished in practice. This is often because there is limited analytical basis for modeling the parameters that the sub-systems have in common. Consequently, attempts to achieve systems integration degenerates to mere conceptual formulations. Sometimes the conceptual formulations work, but most times they do not. Omicini and Papadopoulos (2001) present systems integration as the coordination and management of interactions among entities of a system. The premise of this paper is that a rigorous set-theoretic formulation can help identify the dominant common elements of subsystems, thereby facilitating practical implementation that is amenable to quantitative optimization. The paper explores fundamental set theory and principles for modeling system interactions. A system is often defined as a collection of interrelated elements working together in synergy to produce a composite output that is greater than the sum of the individual outputs of the components. Systems integration is a desired state whereby sub-systems of the system are in proper consonance of operation and input-output interactions. Methodologies pertaining to the various characteristics of integrated systems are available in the literature. Yang et al (2006) present techniques for assessing reliability of integrated systems under time-varying tasks. Chen and Liu (2004) address the time-varying aspect by incorporating fuzzy logic reasoning. Heymann et al (2005) and Zhang et al (2001) considered integration complexity of hybrid systems. As a practical application example, Balkhi (1999) addressed integrated inventory systems with time-varying demand. The approach presented in this paper differs from those existing in the literature by focusing on set-theoretic modeling.

0.1 Theory of Sets and Systems

Set theory is the mathematical concept that governs sets, which represent collections of objects. Sets are defined by the elements contained in the collection and the respective memberships of the elements in the sets. Set theory provides the formalism in which mathematical objects are described. This makes set theory suitable for defining systems integration. In this conception, sets and set membership are fundamental concepts describing points and lines in Euclidean geometry. The properties of sets essential for modeling systems integration include:

- 1. Ordered Pairs
- 2. Relations
- 3. Functions
- 4. Set Cardinality

In this paper, standard set and matrix notations and symbols that are generally available in the literature are used for the mathematical formulations presented.

0.1.1 Ordered Pairs on Sets

The notion of *ordered pair* suggests that if **a** and **b** are sets, then the *unordered pair* $\{a, b\}$ is a set whose elements are exactly a and b. The "order" in which a and b are put together plays no role. Thus, $\{a, b\} = \{b, a\}$. For many applications, we need to pair a and b in a way making possible to "read off" which set comes

"first" and which comes "second." We denote this *ordered pair* of a and b by (a, b); a is the *first coordinate* of the pair (a, b), b is the *second coordinate*. To assess systems integration properties, the ordered pair has to be a set. It should be defined in such a way that two ordered pairs are equal if and only if their first coordinates are equal and their second coordinates are equal. This guarantees in particular that $(a, b) \neq (b, a)$ if $a \neq b$. With this understanding, we can distinguish systems integration from the perspective of "push system" or "pull system" because they may not necessarily be identical.

Basic definition of ordered pairs: The basic definition of ordered pair, generally available in the literature specifies that $(a, b) = \{\{a\}, \{a, b\}\}$. If $a \neq b$, (a, b) has two elements, a singleton $\{a\}$ and an unordered pair $\{a, b\}$. We find the first coordinate by looking at the element of $\{a\}$. The second coordinate is then the other element of $\{a, b\}$. If $a = b$, then $(a, a) = \{\{a\}, \{a, a\}\} = \{\{a\}\}$ has only one element. In any case, it seems obvious that both coordinates can be uniquely "read off" from the set (a, b).

Ordered Pair Theorem: $(a, b) = (a', b')$ if and only if $a = a'$ and $b = b'$.

Proof. If $a = a'$ and $b = b'$, then, of course, $(a, b) = \{\{a\}, \{a, b\}\} = \{\{a'\}, \{a', b'\}\} = (a', b')$. The other implication is more intricate. Let us assume that $\{\{a\}, \{a, b\}\} = \{\{a'\}, \{a', b'\}\}$. If $a \neq b$, $\{a\} = \{a'\}$ and $\{a, b\} = \{a', b'\}$. So, first, $a = a'$ and then $\{a, b\} = \{a, b'\}$ implies $b = b'$. If $a = b$, $\{\{a\}, \{a, a\}\} = \{\{a\}\}$. So $\{a\} = \{a'\}$, $\{a\} = \{a', b'\}$, and we get $a = a' = b'$, so $a = a'$ and $b = b'$ holds in this case also. Ordered pairs can be extended to define *ordered triples:* $(a, b, c) = ((a, b), c)$. Similarly, we can define *ordered quadruples*: $(a, b, c, d) = ((a, b, c), d)$, and so on. These extensions are useful for modeling sub-systems that number more than two.

0.1.2 Set Relations

Set relation is defined in terms of a binary relation, which is determined by specifying all ordered pairs of objects in that relation. **Relation Definitions:** A set **R** is a *binary relation* if all elements of **R** are ordered pairs, i.e., if for any z in R there exist x and y such that $z = (x, y)$. It is customary to write $x\mathbf{R}y$ instead of $(x, y) \in \mathbf{R}$. We say that x *is in relation* **R** *with* y if $x\mathbf{R}y$ holds. The set of all x which are in relation **R** with some y is called the *domain* of **R** and denoted by "Dom **R**." So Dom **R** = $\{x \mid$ there exists y such that $x\mathbf{R}y\}$. Dom **R** is the set of all first coordinates of ordered pairs in **R**. The set of all y such that, for some x, x is in relation **R** with y is called the *range* of **R**, denoted by "ran **R**." So ran **R** = $\{y \mid$ there exists x such that $x\mathbf{R}y\}$. In modeling systems integration parameters, the relations mapped from one sub-system to another become defining characteristics of the integration.

0.1.3 Functions on Sets

A function represents a special type of relation. A function is a rule that assigns to any object a, from the domain of the function, a unique object b, the value of the function at "a." The function implies a relation where every object "a" from the domain is related to precisely one object in the range, namely, to the value of the function at a. This makes it possible to characterize component-to-component relationships for systems integration purposes.

Function Definitions: A binary relation **F** is called a *function* (or *mapping*,

correspondence) if $a\mathbf{F}b_1$ and $a\mathbf{F}b_2$ imply $b_1 = b_2$ for any a, b_1, and b_2. In other words, a binary relation **F** is a function if and only if for every a from Dom **F** there is exactly one b such that $a\mathbf{F}b$. This unique b is called the *value of* **F** *at* a and is denoted $\mathbf{F}(a)$ or $\mathbf{F}_a$. Note that $\mathbf{F}(a)$ is not defined if $a \notin$ Dom **F**. If **F** is a function with Dom **F** = **A** and range **F** $\subseteq$ **B** , we use the notations $\mathbf{F} : \mathbf{A} \mapsto \mathbf{B}$, $< F(a) \mid a \in \mathbf{A} >$, $< \mathbf{F}_a \mid a \in \mathbf{A} >$, $< \mathbf{F}_a >_{a \in \mathbf{A}}$ for the function ***F***. The range of the function **F** can then be denoted $\{\mathbf{F}(a) \mid a \in \mathbf{A}\}$. The Axiom of Extensionality can be applied to functions as follows.

Lemma. Let **F** and **G** be functions. **F** = **G** if and only if Dom **F** = Dom **G** and **F*(x)*** = **G*(x)*** for all $x \in$ Dom **F**. A function f is called *one-to-one* or *injective* if $a_1 \in$ Dom f, $a_2 \in$ Dom f, and $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$. In other words if $a_1 \in$ Dom f, $a_2 \in$ Dom f, and $f(a_1) = f(a_2)$, then $a_1 = a_2$.

0.1.4 Cardinality of Sets

The most obvious property of a set relates to how many elements it contains. It is a fundamental observation that we can define the statement "sets **A** and **B** have the same number of elements" without knowing anything about the elements.

Cardinality Definitions: Sets **A** and **B** have *the same cardinality* if there is a one-to-one function f with domain **A** and range **B**. We denote this by $|\mathbf{A}| = |\mathbf{B}|$. The cardinality of **A** is less than or equal to the cardinality of **B** (notation: $|\mathbf{A}| \leq |\mathbf{B}|$) if there is a one-to-one mapping of **A** into **B**. Notice that $|\mathbf{A}| \leq |\mathbf{B}|$ means that $|\mathbf{A}| = |\mathbf{C}|$ for some subset **C** of **B**. We also write $|\mathbf{A}| < |\mathbf{B}|$ to mean that $|\mathbf{A}| \leq |\mathbf{B}|$ and $|\mathbf{A}| \neq |\mathbf{B}|$, i.e., that there is a one-to-one mapping of **A** onto a subset of **B**, but there is no one-to-one mapping of **A** onto ***B***.

Lemma:

1. If $|\mathbf{A}| \leq |\mathbf{B}|$ and $|\mathbf{A}| = |\mathbf{C}|$, then $|\mathbf{C}| \leq |\mathbf{B}|$.
2. If $|\mathbf{A}| \leq |\boldsymbol{B}|$ and $|\mathbf{B}| = |\mathbf{C}|$, then $|\mathbf{A}| \leq |\mathbf{C}|$.
3. $|\mathbf{A}| \leq |\mathbf{A}|$.
4. If $|\mathbf{A}| \leq |\mathbf{B}|$ and $|\mathbf{B}| \leq |\mathbf{C}|$, then $|\mathbf{A}| \leq |\mathbf{C}|$.

Cantor-Bernstein Theorem: If $|\mathbf{X}| \leq |\mathbf{Y}|$ and $|\mathbf{Y}| \leq |\mathbf{X}|$, then $|\mathbf{X}| = |\mathbf{Y}|$.

I. SET-TO-SYSTEM AND SUBSET-TO-SUBSYSTEM RELATIONSHIPS

Based on the underlying principles of sets and relations, we can now take a system as a set and subsets as subsystems. Each system is represented by *n*-tuple as an extension of an ordered pair. For systems that are describable by a matrix, the eigenmodes, in order of magnitude, will be the *n-th* tuple and we take the dominant modes. This will generate a reduced order representation using singular value decomposition, which is addressed in the next section. Consider the domain of set relations such that if **R** is a relation from set **A** to set **B**, then the domain of **R** is the set of all elements "*a*" belonging to A such that $a\mathbf{R}b$ for some element "*b*" belonging to **B**. That is,

$$Dom(\mathbf{R}) = \{a \in \mathrm{A} | (a,b) \in \mathbf{R} \text{ for some } b \in \mathbf{B}\} \quad (1.1)$$

Consider the two framed sets in Figure 1 (Badiru and Cheung, 2002). Let Frame **A** contain the subsets *a1, a2, a3, a4,* and *a5* while Frame **B** contains the subsets *b1, b2, b3, b4, b5, b6, b7,* and *b8*. Let us define an "*integrating"* relation **Z** from **A** to **B** such that an element *a* belonging to **A** is related to an element b belonging to **B**, if and only if there is a rule in **B** that has *a* as a premise and *b* as a complement.

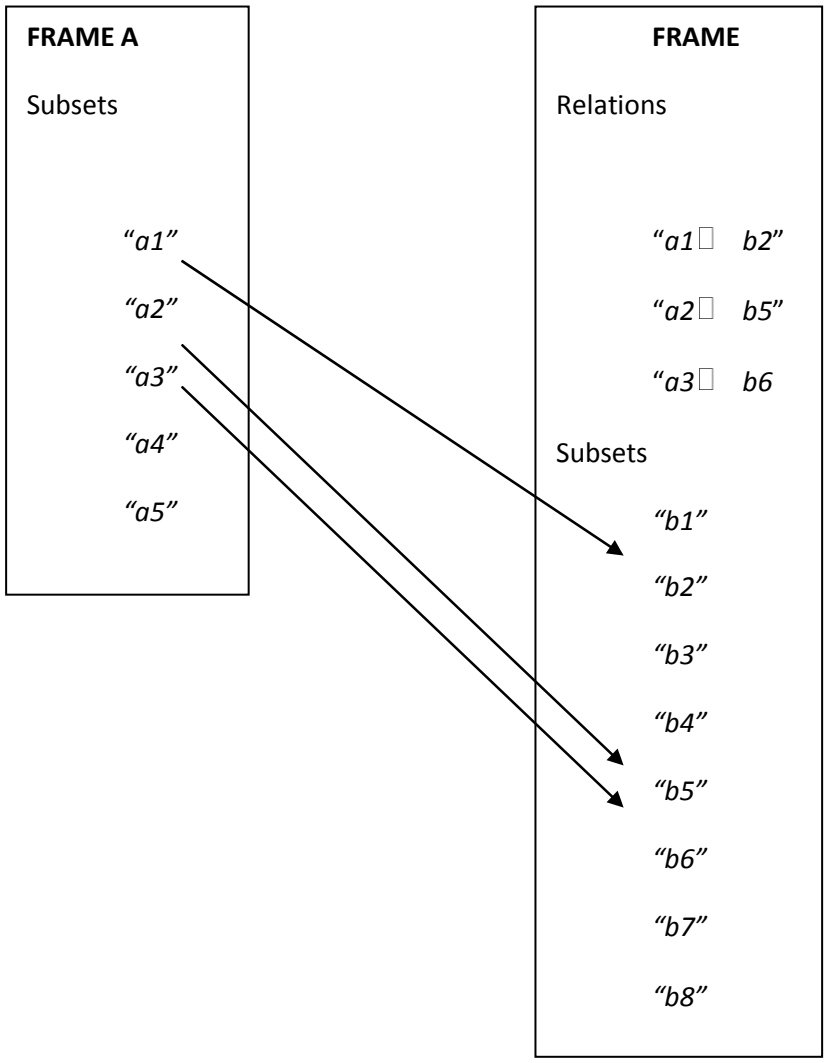

Figure 1. Relations on System Subsets

From Figure 1, we can define the following relationships:

Z = Relation of relevant subset correspondence

Domain of **Z** = {*a1,a2,a3*}

Image of **Z** = {*b2,b5,b6*}

(1.2)

Now suppose only the following relationships exist in Frame **B**:

$$\begin{aligned} a1 &\leftrightarrow b2 \\ a2 &\leftrightarrow b5 \\ a3 &\leftrightarrow b6, \end{aligned} \tag{1.3}$$

where $\leftrightarrow$ represents reciprocity of parameter correspondence. The domain of **Z** is then given by the set:

$$Dom(\mathbf{Z}) = \{a1, a2, a3\}, \tag{1.4}$$

Since the elements a1, a2, and a3 are the only elements of **A** that can successfully integrate with elements in **B**. The image of the relation, **R**, is then defined as:

$$\mathrm{Im}(\mathbf{R}) = \{b \in \mathbf{B} \mid (a,b) \in \mathbf{R} \text{ for some } a \in \mathbf{A}\} \tag{1.5}$$

Thus, the image of the relation **Z** (measure of system integration) is:

$$\mathrm{Im}(\mathbf{Z}) = \{b2, b5, b6\}, \tag{1.6}$$

which corresponds to the set of integrating relations that are effected in Frame **B**. A special kind of relation on integrating sets is parameter mapping in which there is one-to-one correspondence between elements in subsets (subsystems). An identity relation in a subsystem is a relation that integrates every element with itself such that:

$$\mathbf{R} = \{(a,a) \mid a \in \mathbf{A}\}, \tag{1.7}$$

which is a *reflexive* (automatic) property indicating that every subsystem perfectly integrates with itself. In a similar vein, we can define a binary operation on a subsystem, **S**, whereby the Cartesian square **SXS** produces other elements not necessarily in **S**. For example, let **S** be the set of words in a dictionary. An operation may be defined for the process of forming natural language statements or new concatenation of words from the set. The new set of word-statements, although not resident in **S**, do originate from the elements of **S**; thus signifying interrelationships (or integration) of the elements in **S.** As another example, from the set of natural numbers **N**, one could randomly form a sequence of numbers such as 3, 1, 4, 1, and 7. This sequence itself is not in the set **N,** although each term of the sequence is in **N**. Based on the definition of relations on sets, the binary operation approach can, potentially, be used to convey a measure of integration across sub-systems.

1.1 Integration Mapping of Subsets

If the set **C** is a subset of the Cartesian product, **A×B**, of the sets **A** and **B**, then **C** is an integration mapping from **A** to **B** such that for each *a* belonging to **A** there is exactly one *b* belonging to **B** for which (*a, b*) belong to **C**. Thus, the mapping, **C**, is the collection of the elements of **A×B** that have one-to-one correspondence or integration with the elements of **A** and **B**. The elements of the sets **A**, **B**, and **C** are specific parameters of objects contained in the main set of interest. In the word-set example mentioned earlier, **C** = **A×B** since each element *a* of **A**, in integration with each element *b* of **B**, yields a unique (distinct) ordered pair of words. The ordered pair formation can be extended to n-tuple to represent a finite sequence (or ordered list) of elements pulled from multiple systems to form an integrated subset. Thus, n-tuples can be used to mathematically describe integrated objects that consist of specified elements of cooperating systems of dimension *n*.

I.2 Model Reduction Approach

In order to model systems integration, particularly for system of systems, it is helpful

to represent the component systems in a more tractable form. Many alternate representational forms are available in the literature. One approach is to develop representational equations of model reduction of integrated systems. Soong (1977) presents a model reduction of large scale systems using a multivariate linear regression approach. Ibidapo-Obe (1982a) presents the model reduction in another form based on a singular value decomposition approach, which is more efficient computationally. For example, consider an n[th]-order linear system S_1 defined by

$$S_1 : \dot{x} = Ax + B \tag{1.8}$$

where x is the n-dimensional state vector, A is an $n\mathbf{X}n$ system matrix, and u is a p-dimensional input vector. Let z be an m-vector ($m < n$) related to x by

$$z = \mathbf{C}x. \tag{1.9}$$

In model reduction, it is desirable to find an m[th]-order system S_2 described by

$$S_2 : \dot{z} = Fx + Gu \tag{1.10}$$

The $m\mathbf{X}n$ matrix **C** in Eq. (1.9) is the aggregation matrix and S_2 is the aggregated system or the reduced model. It is easy to show that $G = CB$ and that F must satisfy the matrix equation:

$$\mathbf{FC} = \mathbf{CA}. \tag{1.11}$$

S_1, S_2 are relations whilst A, B, C, F, G are sets/subsets

Equation (1.11) defines an over-specified system of equation for the unknown matrix **F**, and hence **F** must be approximated. In Soong (1977), a multivariate linear regression scheme is used to yield a "best" approximation for F in the form

$$\hat{\mathbf{F}} = \mathbf{CAC}^T\left(\mathbf{CC}^T\right)^{-1}, \tag{1.12}$$

where T and -1 denote matrix transpose and matrix inverse, respectively. The rank of **C** is assumed to be m. The result given by Eq. (1.12) is interpreted as a linear, unbiased, minimum-variance estimate of **F** and its form agrees with that given by Aoki (1968), following an ad hoc procedure. In addition, the covariance of $\hat{\mathbf{F}}$ is found to be

$$\mathrm{cov}(vec\hat{F}) = \sigma^2[(CC^T)^{-1} \otimes I_m] \tag{1.12}$$

and it is shown in Soong (1977) that this covariance matrix can be used for model reduction error assessment. In Eq. (1.13), the Kronecker product $\otimes$ and the "vec" operator are defined as

$$P \otimes Q = [P_{ij} \quad Q] \tag{1.13}$$

$$vec(P) = [P_1 \; P_2 \cdots]^T \tag{1.14}$$

where **P** and **Q** are matrices of arbitrary dimensions and P_k is the k[th] column of **P**.

I.3 Singular Value Decomposition

From the computational point of view, it is desirable to circumvent the use of matrix inverses in Eqs. (1.12) and (1.13), particularly for systems having large aggregation matrices. In what follows, this is accomplished through the use of matrix singular value decomposition (SVD), which has found useful application in linear least squares problems (Golub and Reinsch, 1970; Golub and Van Loan, 1979). The SVD concept gives the Moore-Penrose pseudo-inverse of **C** as

$$C^{+} = V\Lambda U^{T} \tag{1.15}$$

where **U** and **V** are unitary matrices whose columns are the eigenvectors of matrices $\mathbf{DD}^T$, and $\mathbf{D}^T\mathbf{D}$, respectively, and

$$\Lambda = \begin{bmatrix} \sigma_1^{-1} & & 0 & & \\ & \sigma_2^{-1} & & & \\ & & \ddots & \vdots & \\ & 0 & & \sigma_m^{-1} & \vdots \\ & \cdots & \cdots & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & 0 \end{bmatrix}_{n\times n}$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m \geq 0$, called singular values, are the nonnegative square roots of the eigenvalues of $\mathbf{D}^T\mathbf{D}$. A discussion of this decomposition and its properties can be found in Stewart (1973). Now, Eq. (1.11) gives

$$\hat{F} = CAC^{+} \tag{1.16}$$

And, using SVD, we can write

$$\hat{F} = CA(V\Lambda U^{T}) \tag{1.17}$$

The matrix **C** can also be written in the form

$$\mathbf{D} = \mathbf{U}\Sigma\,\mathbf{V}^T \tag{1.18}$$

where

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & 0 & & \\ & & \ddots & & \\ & 0 & & \sigma_m & 0 \\ & & & & \ddots \\ & & & & 0 \end{bmatrix}_{nxn}$$

and we have

$$\hat{F} = U\sum V^T AV\Lambda U^T) \tag{1.19}$$

Compared with Eq. (1.12), either Eq. (1.18) or Eq. (1.20) provides a more efficient method of computation for $\hat{\mathbf{F}}$ due to elimination of the matrix inverse. Similarly, advantages are realized in the calculation of $\mathrm{cov}(vec\hat{F})$. Following the SVD scheme,

$$\begin{aligned} \left(DD^T\right)^{-1} &= \left(D^T\right)^{+} D^{+} \\ &= \left(U\Lambda V^T\right)\left(V\Lambda U^T\right) \\ &= \left(U\Lambda^2 U^T\right) \end{aligned} \tag{1.20}$$

Equation (1.13) now takes the form

$$\mathrm{cov}(vec\hat{F}) = \sigma^2[U\Lambda^2 U^T \otimes I_m) \tag{1.21}$$

which is clearly of a simpler structure than Eq. (1.13).

I.4 Sub-System Surface Projection Integrals

Using the concept of system reduction; we consider the projection of one sub-system onto the plane of another sub-system to identify common region, area, or parameters (dominant eigenvalues) for integration. The idea is to identify sub-sets that represent segments (sub-systems) of the systems to be integrated. Fischer-Cripps (2005) presents formulations of line integrals that are adapted for the systems modeling in this section. Consider a surface $z = f(x,y)$ in **Figure 2**. Let a line increment on the curve C in the *xy* plane be given by Δs.

The area of the band formed from the line increment to the surface is given by the product expressed as shown below:

$$f(x, y)\Delta s$$

The total area under the surface along the curve from P1 to P2 represents the sum as Δs goes to zero. That is,

$$\lim_{n\to\infty;\Delta s\to 0}\sum f(x, y)\Delta s \qquad (1.22)$$

This limit is called the line integral of *f(x, y)* along the curve C from P1 to P2. Note that C is used to represent an interval or group of intervals along the curve.

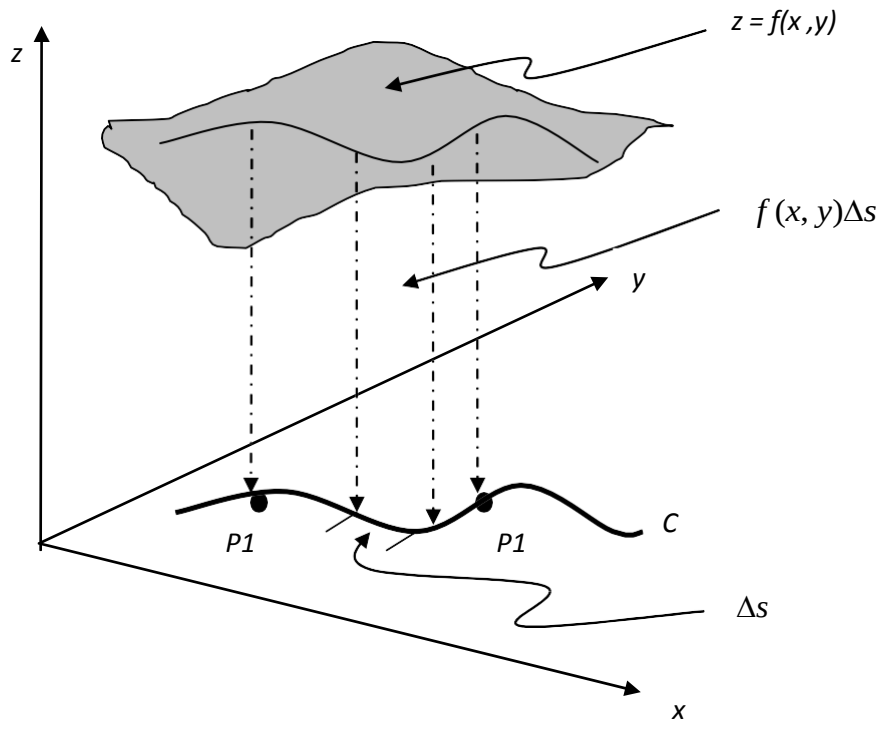

Figure 2. Line Integral of Surface Projection

Referring to Figure 2, the line integral is written as:

$$\int_{P1}^{P2} f(x, y)ds = \int_{C} f(x, y)ds \qquad (1.23)$$

If the curve C can be represented as a **vector function** in terms of the path or arc length parameter *s*, such that:

$$\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j}$$

Then the line integral becomes:

$$\int_{P1}^{P2} f(x(s), y(s))ds \qquad (1.24)$$

If the curve C is given in terms of the parameter *t* such that:

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

Then, we have:

$$\int_{C} f(x, y)ds = \int_{P1}^{P2} f(x(t), y(t))\frac{ds}{dt}dt \qquad (1.25)$$

where $$\frac{ds}{dt} = \sqrt{\frac{dr}{dt}.\frac{dr}{dt}} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

(1.27)

The line integral can be extended to three dimensions where F = *f(x, y, z)*. Then, the line integral is represented as:

$$\int_{C} f(x, y, z)ds \qquad (1.27)$$

It is possible to express the line integral of F(*x, y)* along C with respect to *x* (or *y*) instead of the arc length *s*. This gives the projected area on the xz plane (for Δx) or the yz plane (for Δy). That is, the curve segment Δs projects onto x-axis as Δx and onto y-axis as Δy. Consequently, in this case, the surface *z* may be represented by the function *f(x, y)* in the *x* direction and *g(x, y)* in the *y* direction. The line integral then becomes:

$$\int_{C} F(x, y)ds = \int_{C} f(x, y)dx + \int_{C} g(x, y)dy$$
$$= \int_{C} (f(x, y, z)dx + g(x, y, z)dy) \qquad (1.28)$$

For the general case for three dimensions, the line integral is given by the sum of the projected line integrals and the functions, *f*, *g*, and *h*. Then, we have:

$$\int_{C} F(x, y, z)ds = \int_{C} f(x, y, z)dx + \int_{C} g(x, y, z)dy + \int_{C} h(x, y, z)d$$
$$= \int_{C} (f(x, y, z)dx + g(x, y, z)dy + h(x, y, z)dz \qquad (1.29)$$

The functions *f, g,* and *h* may be components of a vector function **F**, such that:

$$\begin{aligned}\mathbf{F} &= f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k} \\ &= \mathrm{F}_1\mathbf{i} + \mathrm{F}_2\mathbf{j} + \mathrm{F}_3\mathbf{k}\end{aligned} \tag{1.30}$$

Now, we have:

$$F_1dx + F_2dy + F_3dz = \left(F_1\frac{dx}{ds} + F_2\frac{dy}{ds} + F_3\frac{dz}{ds}\right)ds = \left(\mathbf{F}.\frac{d\mathbf{r}}{ds}\right)ds \tag{1.31}$$

where:

$$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} \tag{1.32}$$

and

$$\frac{d\mathbf{r}}{ds} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k}$$

Consequently, the line integral $\int_C \mathbf{F}.\frac{d\mathbf{r}}{ds}ds$ can be written as $\int_C \mathbf{F}.d\mathbf{r}$, which is referred to as the circulation of **F** around C. In general, the value of the line integral depends upon the path of integration. If, however, the vector field **F** can be represented by:

$$\mathbf{F} = \nabla\phi, \tag{1.33}$$

Then the line integral is independent of the path. In this case, ϕ is called the scalar potential. The field **F** is said to be a conservative field and $d\phi$ is called an exact differential. Now, we have

$$\begin{aligned}\int \mathbf{F}.dr &= \int_{P1}^{P2} \nabla\phi.d\mathbf{r} \\ &= \int_{P1}^{P2} \nabla\phi.(dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) \\ &= \int_{P1}^{P2} \left|\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right|.(dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) \\ &= \int_{P1}^{P2} \left|\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right| \\ &= \int_{P1}^{P2} d\phi \\ &= \phi(x_2, y_2, z_2) - \phi(x_1, y_1, z_1)\end{aligned} \tag{1.35}$$

The integral depends only on the end points *P1* and *P2* and not on the path joining them. This is verified by the fact that the form or equation of the curve C is not defined in the above formulations. If the path of integration of C is a closed curve, and if **F** is conservative, then we have:

$$\oint_C \mathbf{F}.d\mathbf{r} = 0, \tag{1.34}$$

where the circle in the integral sign denotes integration around a closed curve. For systems integration application, the area of integration can be used to represent the magnitude of overlap between sub-systems.

I.5 Set Projection and System Overlap

Systems overlap for integration purposes can further be represented as projection integrals by considering areas bounded by the common elements of sub-systems. In Figure 3, the projection of a flat plane onto the first quadrant is represented as area *A* while Figure 4 shows the projection on an inclined plane as area B.

$$A = \iint_{A_y A_x} z(x, y)dydx \qquad (1.35)$$

$$B = \iint_{B_y B_x} z(x, y)dydx \qquad (1.36)$$

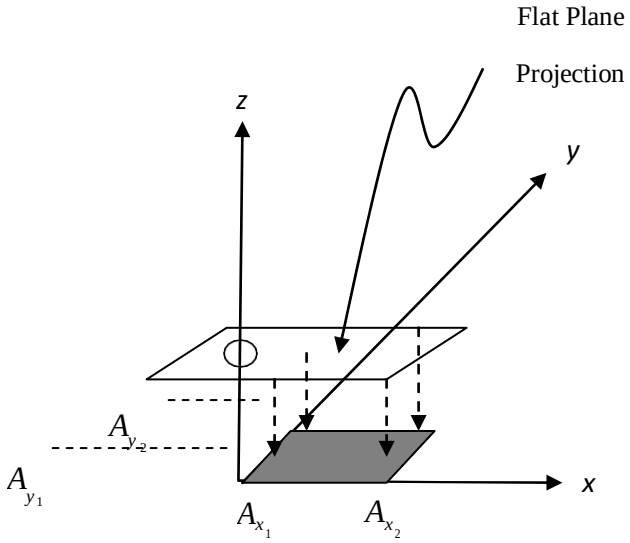

Figure 3. Flat Plane Projection

The net projection encompassing the overlap of A and B is represented as area C, shown in Figure 5, and computed as:

$$C = \iint_{C_y C_x} z(x, y)dydx \qquad (1.37)$$

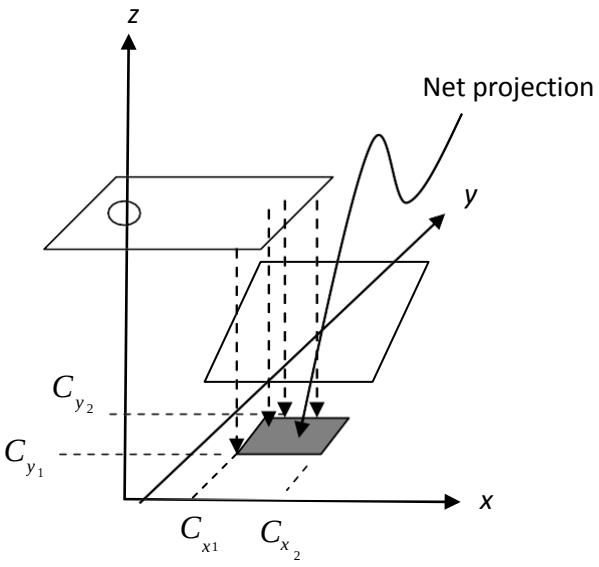

Figure 5. Net projection area

II. TIME-VARIANT SYSTEMS INTEGRATION

Time-variant systems pose additional complexity to systems integration. This can be addressed by stochastic techniques. The problem of stochastic estimation in non-linear systems has been formulated using different mathematical models and techniques such as the stochastic differential equation approach and function-space methods. Ibidapo-Obe (1982b) presents a scheme, based on eigenmode expansion, for the estimation of system responses in distributed parameter systems. Although some of the expositions demonstrate organic generality of the proposed non-linear structures, the techniques available for the analysis and synthesis of these structures for stochastic processes do not yield solutions for the generalized models. An enhanced technique for optimal estimation of system parameters in a stochastic environment with non-linear sensors based on the eigenmode representation of generalized displacements and orthogonal expansion of the conditional joint probability distribution is now illustrated. A linear dynamical system shows the correspondence between the innovations process, the observations process and the integrated system response. The proposed algorithm which may be applied to several civil engineering and space structures, best described by the distributed parameter equation

$$m(x)u_{tt}(x,t)+Bu_t(x,t)+Cu(x,t)=F_0(x,t) \tag{2.1}$$

with the initial conditions

$$u(x,t_0)=u^0 \quad \text{and} \quad u_t(x,t_0)=u_t^0 \tag{2.2}$$

where $m(x)$ is the mass per unit length (assumed unity); $u(x,t)$ the vector of generalized displacements; **B** and **C** are in general spatial differential operators representing the damping and restoring forces respectively; and $\mathbf{F}_0(x, t)$ is the external impressed forces (wind loads, earthquake excitations, etc.).

II.1 Modal Canonical Representation

Let the displacement be expressed in terms of structural dominant mode shapes $\{\phi(x)\}_1^N$ with associated frequencies $\{w\}_{i\,1}^N$

$$u(x,t)=\sum_1^N u_i(t)\phi_i(x); \quad F_0(x,t)=\sum_1^N F_i(t)\phi_i(t) \tag{2.3}$$

so that

$$u(x,t)=U^T(t)\Phi(x)$$

and (2.4)

$$F_0(x,y)=F_0^T(t)\Phi(x)$$

where

$$\mathbf{U}=(u_1, u_2 \ldots, u_N)^T, \quad \mathbf{F}_0=(\mathbf{F}_1^0, \mathbf{F}_2^0, \ldots, \mathbf{F}_N^0)^T \tag{2.5}$$

and

$$\Phi=(\phi_1, \phi_2, \cdots, \phi_N)^T \tag{2.6}$$

Substituting eqn. (2.4) to eqn. (2.1) gives

$$\ddot{U}^T(t)\Phi(x)+B\dot{U}^T(t)\Phi(x)+CU^T(t)\Phi(x)=F_0^T(t)\Phi(x) \tag{2.7}$$

thus obtaining the i^{th} mode amplitude equation

$$\ddot{u}_i(t)+2\xi_i\omega_i\dot{u}_i(t)+\omega_i^2u_i(t)=F_i^0(t) \tag{2.8}$$

where $C\phi_i=\omega_i^2\phi_i$ and $B\phi_i=2\xi_i\omega_i\phi_i$ with appropriate initial conditions.

Equation (2.8) can now be put in the state space form

$$\dot{q}(t) = Aq(t) + F \tag{2.9}$$

such that

$$q = (q_1, q_2)^T \tag{2.10}$$

with

$$q_1 = u_i, \quad q_2 = \dot{u}_i$$
$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -2\xi\omega \end{pmatrix} \tag{2.11}$$

and

$$\mathbf{F} = \left(0, \mathbf{F}^{i0}\right)^T \tag{2.12}$$

The sensors are assumed to have the form

$$y(t) = z(t) + v(t); \quad z(t) = h[q(s); s \le t] \tag{2.13}$$

where

$$E\{v(t)\} = 0;\ E\{v(t)v^T(s)\} = R(t)\delta(t-s);\ E\{z(t)\} = 0;\ |z(t)| \le M < \infty$$

for all t; $\int E\{z(t)z^T(t)\}dt < \infty$; $E\{v(t)F^T\} = 0$

and h is a functional of $q(s)$. It is further assumed that the forcing function F is additively (preferably Gaussian) random.

II.2 Canonical Estimation Procedure

The least squares estimate of $q(t)$ is

$$\hat{q}(t|\tau) = E\{q(\tau)|y(s), \quad 0 \le s \le \tau\} \tag{2.14}$$

Assuming that the property of "causal equivalence" holds, the sensor equations may be transformed into a white Gaussian process $v(t)$, called the strict sense innovations process with a simpler structure so that eqn. (2.14) can now be put in the form

$$\hat{q}(t|\tau) = \int_0^\tau E\{q(t)v^T(s)|v(\sigma), \quad 0 \le \sigma \le s\}\, v(s)ds \tag{2.15}$$

where $v(t) = y(t) - \hat{z}(t|t), 0 \le t < \tau$ and $\int$ is the *Itô* (1951) integral. A differential structure for eqn. (2.15) is obtained as

$$\dot{\hat{q}}(t|t) + A\hat{q}(t|t) + K(t)v(t) \tag{2.16}$$

where

$$K(t) = E\{\hat{q}(t)v^T(t)|v(\sigma), \quad 0 \le \sigma \le t\} \tag{2.17}$$

Conditional joint probability distribution functions are required to obtain explicit solution to the stochastic differential equation (2.16). A method for the evaluation of the probability densities is obtained in the form of a partition theorem (Lainiotis 1971).

$$p(q|f^t) = \Lambda(t|q)p(q)/\int p(q)\Lambda(t|q)dq \tag{2.18}$$

with $f^t = \{y(t)\}_0^t$ and

$$\Lambda(t|q) = \exp\left\{\int_0^t \hat{h}(s,q)R^{-1}(s)y(s)ds - \frac{1}{2}\int_0^t h(s,q)^{R^{-1}}ds\right\} \tag{2.19}$$

where

$$\hat{h}(s,q) = E\{h(s,q)|f^s\} \tag{2.20}$$

and $p(q_0)$ is the priori density function. Equation (2.18) can be approximated by expanding the distributions $A(t|q)$ in a Volterra power series (Barrett and Lampard 1955) with functional terms

$$\Lambda(t|q) = \sum_{i=0}^{\infty} \Lambda_i(t|q) \tag{2.21}$$

where

$$\Lambda(t|q)=\frac{1}{i!}\int_0^{\infty} d\tau_1\cdots\int_0^{\infty} d\tau_i\, g_i(\tau_1,\tau_2,\cdots\tau_i)\prod_{k=1}^{i} v(t-\tau_k) \quad (2.22)$$

such that $g_i(...)$, $i = 1, ..., n$ are the integral kernels describing the system and $v(\cdot)$ is the innovations process. It has been assumed that in order to satisfy the requirements of physical realizability, the kernels are zero for any argument less than zero. The kernels can be identified in the following manner:

(i) First order kernel: $g_1(t)$

Let $v(t)=A\delta_0(t)$ be an impulse of strength A so that

$$\Lambda(t|q)= Ag_1(t)+ A^2 g_2(t,t)+ A^3 g_3(t.t,t)+ ... \quad (2.23)$$

and

$$g_1(t_1) = \left.\frac{\partial \Lambda(t_1|q)}{\partial A}\right|_{A=0} \quad (2.24)$$

(ii) Second order kernel: $g_2(t,t)$

Let $P(t) = A\delta_0(t)+ B\delta_0(t+\tau)$ be two impulses at t and $t + t$ of strength A and B respectively where A and B are specified constants as above ; then

$$\Lambda(t|\theta)= Ag_1(t)+ Bg_1(t+\tau)+ A^2 g_2(t,t)+ 2ABg_2(t,t+\tau)+ B^2 g_2(t+\tau,t+\tau)+\cdots$$

and the second order kernel

$$g_2(t,\, t+\tau) = \left.\frac{1}{2}\frac{\partial^2\Lambda(t_1|q)}{\partial A\partial B}\right|_{\substack{A=0\\B=0}} \quad (2.25)$$

Higher order kernels can be computed in similar manners. The system state estimates can now be obtained as

$$\hat{q}(t|t)= J(t|t_0)\hat{q}(t|t_0)+\int_{t_0}^{t} J(t|\tau)k(\tau)v(\tau)d\tau \quad (2.26)$$

where

$$J(t|t_0)= \exp\{A(t - t_0)\} \quad (2.27)$$

represents the transition/fundamental matrix; $k(\tau)$ having been approximated using the eqns. (2.18) and (2.21).

II.3 Dynamic System Example

An hypothetical linear dynamical system is solved below, using the Kalman filtering equations, to illustrate the close correlation (in a non-statistical sense) between the observations process, the innovations process and the systems state. It has been assumed that the distributed parameter system is reducible to a linear ordinary differential equation system as follows

System dynamics:

$$\dot{x}(t)= 0.5x(t)+u(t), \quad x(0)=0 \quad (2.28)$$

Observation process:

$$y(t)= x(t)+ v(t) \quad (2.29)$$

Figure 6 represents the observations process, which is simply taken here as the system response plus a white Gaussian noise. Figure 7 is the innovations representation. Figure 8 is the innovations auto-correlation, which indicates that the process is a white noise. Finally, Fig. 9 shows the system state $x(t)$. An approximation technique becomes imperative in the non-linear case since the innovations sequence generated is not a true innovations process and hence a scheme for adaptive improvement has to be employed.

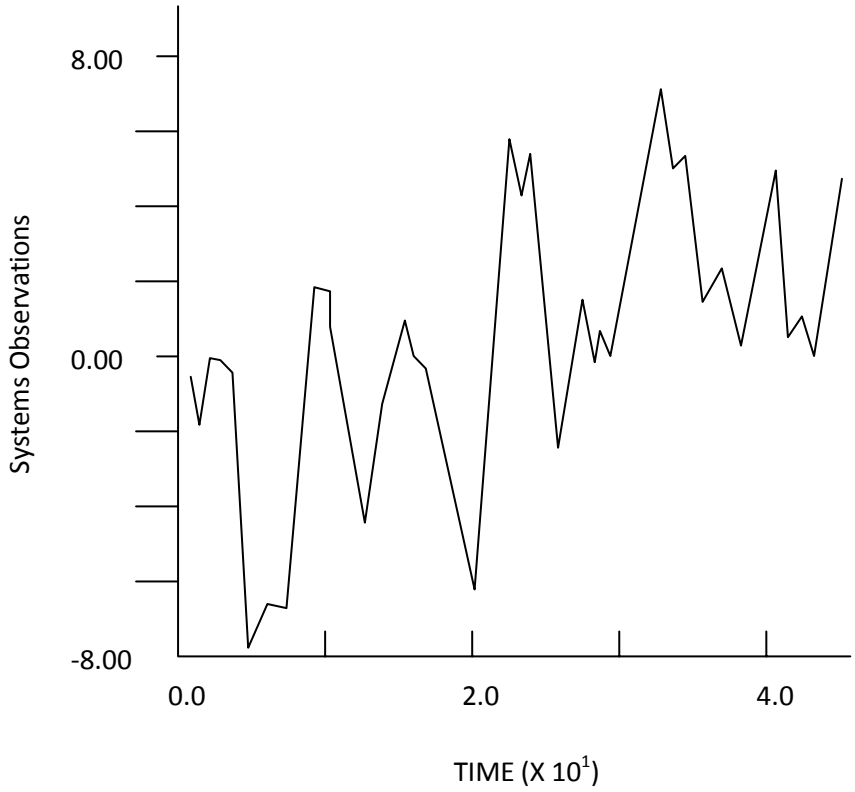

Figure 6. Systems Observation Process

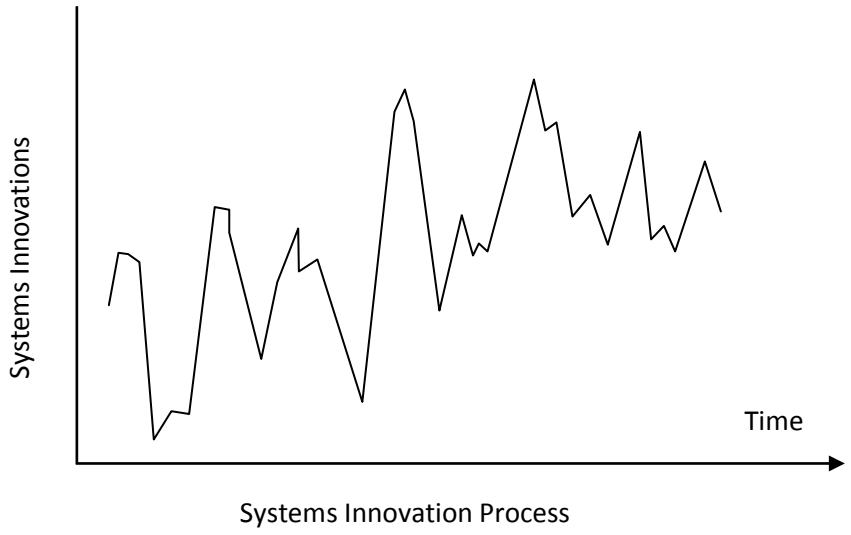

Figure 7. Systems Innovation Process

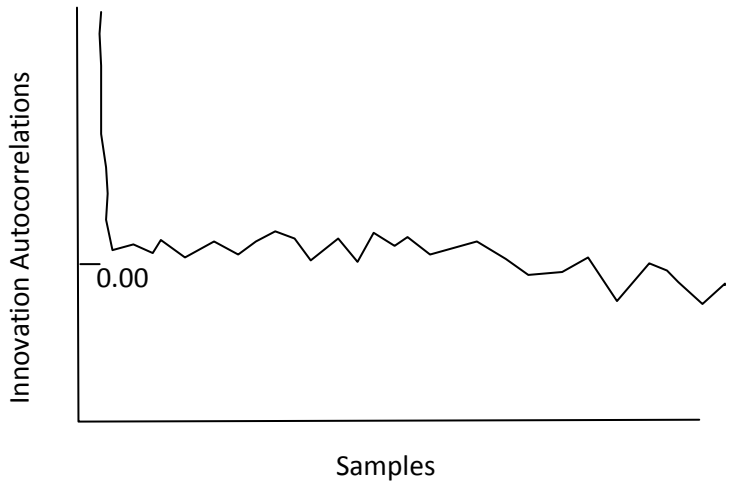

Figure 8. Autocorrelation for innovations process

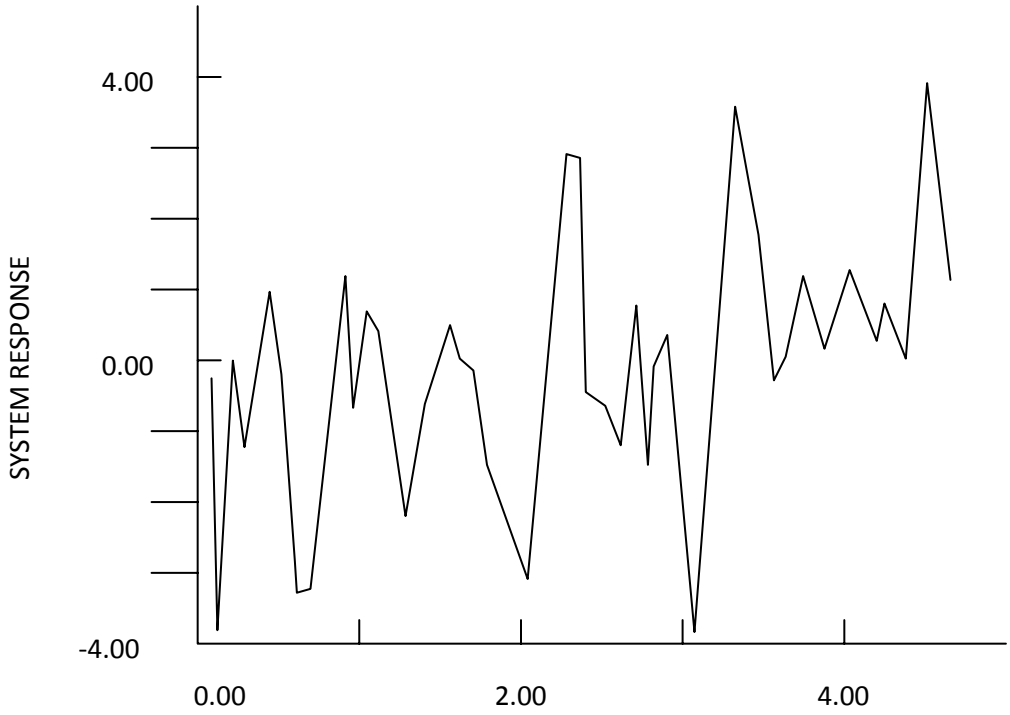

Figure 9. System Response

III. CONCLUSIONS

Systems integration is of general interest as an avenue to enhance organizational cohesion and achieve operational efficiency. However, the process of integrating sub-systems of a massively complex system is very intractable. This is because many underlying factors impinge upon the efficacy of an integrated system. Such factors may include technological, human, psychological, ethical, contractual, economic, and time-dependent issues pertinent for each sub-system to be integrated. So far, there is no quantitative modeling approach to assess the degree of systems integration. The set-theoretic approach presented in this paper attempts to fill that void. It represents the first step to further research in developing quantitative measures for systems integration. In such further research, one must consider various parameters that characterize a system as may be deterministic, stochastic, dynamic, static, continuous, or discrete so that quantitative modeling of integration can be applied. Simulation modeling can also be developed for assessing integration potential for systems involving several components that are too intractable to study analytically.

REFERENCES

[1] Aoki, M., "Control of large scale dynamic systems by aggregation," *IEEE Trans. Automat. Control*, AC-13, 1968, pp. 246-253.

[2] Badiru, Adedeji B. and John Cheung, Fuzzy Engineering Expert Systems with Neural Network Applications, John Wiley & Sons, New York, 2002.

[3] Balkhi, Z. T., "On the global optimal solution to an integrated inventory system with general time-varying demand, production, and deterioration rates," *Journal of Operational Research Society*, 1999, Vol. 114, pp. 29-37.

[4] Barrett, J. F., and Lampard, D. G., "An expansion for some second-order probability distributions and its application to noise problems," *I. R. E. Transactions on Information Theory*, Vol. 1, 1955, pp. 10-15.

[5] Chen, B. and X. Liu, "Reliable control design of fuzzy dynamic systems with time varying delay," *Fuzzy Sets and Systems*, Vol. 146, 2004, pp. 349-374.

[6] Fischer-Cripps, A. C., The Mathematics Companion, Institute of Physics Publishing, Philadelphia, PA, 2005.

[7] Golub, G. H. and C. Reinsch, "Singular value decomposition and least squares solutions," *Numer. Math*. 14, 1970, 403-420.

[8] Golub, G. H., and C. Van Loan. Total least squares, in "Smoothing Techniques for Curve Estimation," T. Gasser and M. Rosenblatt, Eds., pp. 62-76, Springer Verlag, New York, 1979.

[9] Heymann, M.; F. Lin; G. Meyer; and S. Resmerita, "Analysis of zeno behaviors in a class of hybrid systems," *IEEE Transactions on Automatic Control*, Vol. 50, 2005, pp. 376-383.

[10] Ibidapo-Obe, O., "A Note on Model Reduction of Large Scale Systems," *Journal of Mathematical Analysis and Applications*, Vol. 90, 1982a, pp. 480-483.
[11] Ibidapo-Obe, O., "A Modal Approach for Estimation in Distributed Parameter Systems," *International Journal of Systems Science*, Vol. 13, No. 10, 1982b, pp. 1165-1170.
[12] Ito, K., "Multiple Wiener integral," *Journal of Mathematical Society of Japan*, Vol. 3, 1951, pp. 157-169.
[13] Laintiotis, D. G., "Optimal non-linear estimation," *International Journal of Control*, Vol. 14, Issue 6, 1971, pp. 1137-1148.
[14] Omicini, Andrea and George A. Papadopoulos, "Coordination as a Paradigm for Systems Integration," *Journal of Systems Integration*, Vol. 10, 2001, pp. 103-105.
[15] Soong, T. T., "On model reduction of large scale systems**,"** *J. Math. Anal. Appl*. 60, 1977, 477-482
[16] Stewart, G. W., Introduction to Matrix Computations, Academic Press, New York, 1973.
[17] Yang, S. X. Zhengrong, C. Quingwei, and H. Weili, "Robust reliable control of switched uncertain systems with time-varying delay," *International Journal of Systems Science*, Vol. 37, No. 15, 2006, pp. 1077-1087.
[18] Zhang, K. J. Johansson; J. Lygeros; and S. Sastry, "Zeno hybrid systems," *International Journal of Robust and Nonlinear Control*, Vol. 11, 2001, pp. 435-451.